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ABSTRACT

The focus of this research is to describe children's ways of quantifying length and perimeter in response to an open-ended reasoning task. Twenty-five children from grade 2 through grade 10 were given two fixed perimeter tasks in a structured interview setting. A 1997 framework proposed by Clements, Battista, Sarama, Swaminathan, and McMillen was used for the study. Children's solution strategies suggested a modified framework consisting of levels 1a, 1b, 2a, 2b, 3a and 3b, where the distinction of 3a and 3b emerged as a contrast between dynamic and static ways of coordinating and sequencing cases of measured shapes. Although more advanced strategies appeared among older children, some of the younger children nonetheless exhibited level 3 strategies and conversely, some grade 10 students seemed limited to level 2b strategies. The analysis also addresses the students' ways of coordinating representations of space and number, and ways of evaluating boundary cases. This study supports and elaborates upon previous studies based on a classroom curriculum implementation, and on teaching experiments, providing a broad account of developmental levels of knowledge for path length. The development of children's spatial reasoning has recently gained emphasis within the mathematics curriculum, for the U.S. Measurement system and geometry have been identified as two of five content-focused standards for K-12 instruction (NCTM, 2000). Furthermore, reasoning and proof retains a prominent position as one of the five process-focused standards, emphasizing children's ways of explaining and describing their responses to substantial tasks across the curriculum. Measurement and geometry tasks provide critical sites for helping children engage in reasoning toward proof. As the curriculum is shifted to emphasize measurement, especially in the first two of the four K-12 grade bands, there is a concurrent need to describe the increasingly sophisticated ways of measuring how children are expected to develop. If teachers are helped in connecting their own knowledge of children's thinking and strategies for measurement to instructional decisions, the expectation that teachers will provide appropriate sequences of instructional activities, ask relevant questions, and promote mathematically rich discussions. (Contains 44 references and 11 figures.) (Author/ASK)

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Children's Developing Knowledge of Perimeter Measurement in Elementary, Middle, and High School

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ABSTRACT: The focus of this research is to describe children's ways of quantifying length and perimeter in response to an open-ended reasoning task. Twenty-five children from grade 2 through grade 10 were given two fixed perimeter tasks in a structured interview setting. We used a framework proposed by (Clements, Battista, Sarama, Swaminathan, & McMillen, 1997). Children's solution strategies suggested a modified framework consisting of levels 1a, 1b, 2a, 2b, 3a and 3b, where the distinction of 3a and 3b emerged as a contrast between dynamic and static ways of coordinating and sequencing cases of measured shapes. Although more advanced strategies appeared among older children, some of the younger children nonetheless exhibited level 3 strategies and conversely, some grade 10 students seemed limited to level 2b strategies. Our analysis also addresses the students' ways of coordinating representations of space and number, and ways of evaluating boundary cases. This study supports and elaborates upon previous studies based on a classroom curriculum implementation, and on teaching experiments, providing a broad account of developmental levels of knowledge for path length. The development of children's spatial reasoning has recently gained emphasis within the mathematics curriculum for the U.S. Measurement and geometry have been identified as two of five content-focused standards for K-12 instruction (NCTM, 2000). Furthermore, reasoning and proof retains a prominent position as one of the five process-focused standards, emphasizing children's ways of explaining and describing their responses to substantial tasks across the curriculum. Measurement and geometry tasks provide critical sites for helping children engage in reasoning toward proof. As the curriculum is shifted to emphasize measurement, especially in the first two of the four K-12 grade bands, there is a concurrent need to describe the increasingly sophisticated ways of measuring we expect children to develop. If we help teachers connect their own knowledge of children's thinking and strategies for measurement to instructional decisions, we can expect teachers to provide appropriate sequences of instructional activities, ask relevant questions and promote mathematically rich discussions (Jacobson & Lehrer, 2000; Lehrer & Chazan, 1998).

We must provide careful, comprehensive accounts of children's strategies for handling and understanding measurement, including both productive and unproductive strategies, and describe plausible trajectories for children's reasoning about length. In this way, teachers can identify typical kinds of reasoning, recognize common misconceptions and provide substantive tasks to promote increasingly abstract and mathematically coherent ways of reasoning about measurement (Barrett & Clements, (submitted for publication)). This paper describes one such trajectory, elaborating on children's reasoning and strategies as they encounter a task on measuring perimeter.

Measurement has been described as "the assignment of a numerical value to an attribute of an object" (Mathematics, 2000, p. 44). It is a relation between numerical quantity and spatial extent (Bishop, 1983). In this way, measurement is an aspect of the mathematics curriculum that requires children to integrate their number scheme and their spatial scheme as they explore, hypothesize, test and draw conclusions in response to substantial tasks. In describing children's ways of measuring path length and perimeter, researchers have identified point counting as a strategy that is only partially successful. The suggestion that children isolate points as they measure (Cannon, 1992) has been used to explain children's struggles to draw a specific length line segment with a ruler (Kamii, 1995; Lehrer, Jacobson, Kemney, & Strom, 1999; Nitabach & Lehrer, 1996), and to explain children's ways of miscounting as they find perimeter for a region of a tile floor described by a subset of tiles in a rectangular arrangement (Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998; Schifter & O'Brien, 1997). While it is apparent that children attend to points more than to the spaces between hash marks and points, it is also apparent that children's ways of quantifying perimeter as a measure

of the path around a shape are not always explainable as point counting (Barrett & Clements, 1998; Barrett & Clements, (submitted for publication); Steffe, 1991).

Recent investigations into children's reasoning about linear space suggest children gain understanding and consistent strategies in several ways: as they learn to connect counting actions, inscribed drawings and verbal accounts of the actions in terms of the drawn representations (Lehrer et al., 1999; Meira, 1995); as they relate their own motions to a drawing representing a path they have traversed (Clements et al., 1997); and later as they relate their own motions to both a drawing and to an abstracted image of the linear object they are measuring (Barrett & Clements, (submitted for publication); McClain, Cobb, Gravemeijer, & Estes, 1999; Stephan, Cobb, Gravemeijer, & McClain, 1998, April). In their curriculum-development study of 3rd-grade children's developing knowledge of path length, Clements et al. (1997) suggested that children elaborate their use of drawings by moving, segmenting and drawing to complement their own knowledge of path length and to support evolving proportional notions about linear space. These researchers highlight the complex interactions between children's motions along paths, their eye movements along perceptible line segments or paths, and the internal representations of paths as potentially dynamic images. They observed three distinctive strategies among the third-grade children in classrooms they studied: 1) visual guessing of numbers to assign length, 2) making hash marks and counting marks to find length, 3) imposing an imagistic "ruler" and following proportional reasoning to establish length.

This set of three strategies was later hypothesized as a framework describing fourth-grade children's increasingly abstract knowledge of path length as a hierarchical set of three levels of understanding and action (Barrett & Clements, (submitted for

publication)). This latter study found some children who counted hash marks without relating them to units of length and others who used the marks to count a collection of units. This finding suggested sub-classifications of Level 2a and Level 2b to distinguish the two groups of students who used hash marks. While this study supported the framework as a description at the middle level, it did not establish the validity of the hierarchical account at either extreme of the levels framework. Further work is needed to examine the strategies of both less-sophisticated (Level 1) and more-sophisticated students (Level 3). By establishing a framework account of children's strategies for length, we may then attempt to relate these patterns to a broad cognitive account of children's understanding of geometry, of space, and of number.

Thus, the connection between children's broad developmental patterns of reasoning for geometry and space, and the specific trajectory for concepts of path length and perimeter measurement requires further study. How do children gain increasingly complex, abstract, coherent knowledge of length as they advance from early elementary, through upper-elementary, into middle school, and on into secondary grades? How do children engage increasingly effective and consistent ways of measuring path length as they move through these school years?

The present study focuses on developmental patterns as they bear on the growth of measurement strategies and knowledge: we examine how children learn path measurement as they gain ever-increasing abstractions for geometry and mature in their knowledge of number (including proportional reasoning). Here we follow a "mapping theory" of cognition positing a correspondence relation between the figural, tangible world of perceptions and impression, as it can be mapped to an internal world of thought and abstract-verbal or abstract-imagistic knowledge of patterns and selected

aspects from the external world (Miera, 1995). We also employ Fischbein's theory of figural concepts as a point of intersection to the "activity theory" promoted by Meira as a complementary account of children's cognitive processing with geometric objects of thought (Chiu, 1996; Ferrari, 1992; Fischbein, 1993; Fishbein, Lewis, & Keiffer, 1972; Gagatsis & Patronis, 1990).

Furthermore, we examine a specific foundation of measurement knowledge, something Clements (1999) has described as "measurement sense". Thus, length measurement sense consists in the development of spatial reasoning as a coordination of spatial reasoning and numerical reasoning for measuring along a single dimensional path. By attending to children's ways of explaining, their ways of defending their explanations, and by tracing their actions and narratives in pursuing a comprehensive solution to a fixed perimeter task, we show how children coordinate numerical and spatial realms of thinking to investigate a content-focused question on length measurement. It is precisely because of this intersection of these distinct sets of actions and procedures—one for arithmetic and the other for geometry figures--that we find a basis for describing children's developing ways of coordinating space and number; we describe children's responses to a structured task that provokes tension between merely spatial and merely numerical schemes for mathematics.

In summary, we focus on two questions in our study:

1. Does the 3-level account of children's knowledge of length account for the range and breadth of development as children advance from early elementary, through upper-elementary, into middle school, and on into secondary grades?

2. How do children engage their increasingly coordinated knowledge of number and space, their increasingly connected representations for geometric shapes, and their

developing knowledge of measurement to address open questions relating perimeter and side length for polygons?

DESIGN

Participants:

Twenty-five children from Western New York State were interviewed for this study. The children ranged in age from grades 2 through 10: 9 children were in grades 2-3; 10 children were in grades 5-6; and 7 children were in grades 9-10. The children were drawn to reflect a variety of socio-economic settings: including children in grade 2 and grade 5 attending an urban elementary school, children in grades 2-3 attending a rural elementary school, children in grades 5-6 attending a suburban middle school, and children in grades 9-10 attending a suburban secondary school.

Procedures:

We employed a structured, task-based clinical interview addressing children's capacity to find, describe, and elaborate cases of rectangles and triangles having a given perimeter measure. We sought to develop a task that motivated concurrent numerical and spatial reasoning. By asking children to find all cases of a shape with a fixed perimeter we expected some children to exhibit coordinated numerical and spatial reasoning sequences to justify their claims.

The interviews were based on a protocol developed through a set of pilot interviews across all grade levels (see Barrett & Clements, (submitted for publication) for an appendix and further account of this protocol). Five students participated in the pilot round of interviews and also during the formal round of interviewing. For the pilot interviews, the students were asked to examine and describe the set of all possible

triangles having a perimeter of exactly 120 units, and later, of all possible rectangles by making drawings and labeling the parts of the drawing to show the length of each side of a figure as well as its perimeter measure. The magnitude of the number 120 proved unmanageable for many of the young students during pilot work with the task, and the use of drawings proved too difficult for them. Thus, the interview was changed to allow use of a flexible object that all ages of children could form into triangle or rectangle shapes. The object, a flexible, plastic straw measuring 48 cm in length was notched at 2 cm intervals to produce the image of a linear object partitioned into 24 sub-segments. The use of the straw allowed younger children access to the task and supported their use of drawings to record cases of triangles or rectangles they found with the straw.

Furthermore, the research team revised the interview format by expanding the protocols to provide a consistent plan for prompting students in keeping with their evident achievements or struggles at describing sets of figures with a perimeter of 24 units. Each interview began with a discussion of the length of a short section of the straw. This allowed the interviewer and the student to agree on a way of finding length for a sub-length along the straw (children in grades 2 through 5 rarely isolated the units; they were more often attending to notches and not to the spaces defined between the notches). Then the child was asked the following two questions:

Rectangle task

QUESTION 1. Explore the set of rectangles with a perimeter (total length of all sides) of 24. Draw and label several rectangles.

QUESTION 2. Describe the whole set of rectangles with this perimeter. How many are there altogether? Can you explain your answer? Can you prove that you found all of the rectangles having a perimeter of 24?

After addressing these questions regarding rectangles, the questions were restated in terms of triangles. Children were encouraged throughout both portions of the interview to make drawings as "records of their work", including labels for each side length and for perimeter.

While some studies examine students' drawings as a means of discovering their thinking, we followed Miera (1995) who describes children's drawings and figures as a "site" for mathematical reasoning. This perspective required us to focus our discussion around the development of sequences of drawings not only to find out children's thinking, but also to promote their thinking. Thus, the role of the interviewer was not merely that of observer, but also as participant-observer within the prescribed structure of the protocol (Goldin, 1999; also, see: Zang, 1994).

Data Analysis:

We pursued a generative stance as described by Clement (1999) to elaborate on developmental trends in children's spatial reasoning related to length measurement. As described elsewhere, (Barrett & Jones, in preparation) children's strategies for measuring length often range widely across apparent levels of abstraction and sophistication, although there is a "central tendency" that typifies a child at any stage of their development of measurement concepts. Thus, we sought to identify the most salient strategy level for each child and then we sought to examine the broad range of strategies exhibited, and to identify any correlation between strategy and age. We also considered whether the children might exhibit a pattern of strategy development that would be correlated with some factor besides age. Thus, we compared the observed strategy types with those identified in previous studies (Strauss & Corbin, 1990).

We examined the children's mathematics and their strategies for measuring length by analyzing three sources of data: verbal interactions with the clinical interviewer, gestures and movements with the plastic straws and inscribed drawings. In addition, we referred to field notes and reflective notes written by the researcher/interviewer (the first author) at the time of each interview. We characterized each child along the three-level rubric mentioned earlier (Barrett & Clements, (submitted for publication); Clements et al., 1997). At least 2 team members categorized each videotaped interview. No disagreements extended past an adjacent level (from the levels at: 1, 2a, 2b, and 3). When team members disagreed (as we did on 4 of the 25 interviews), another team member joined the analysis and the case was reviewed until a consensus was reached. We met regularly to discuss specific findings for each child, and later for grade-level groups of children. We also characterized each child's way of representing path length and perimeter and examined each child's forms of argument or exploration as they sought to exhaust the set of cases that satisfy the fixed perimeter condition (either for triangles or rectangles).

As we analyze the work of the children in this study, we adopt the following system for referring to a particular child: The first letter of their name is given (and a second letter is sometimes needed to distinguish children) next to a number that describes their grade level. For example, a child named Susan in the third grade might become S3 (or perhaps Su3 if there were another child named Sandy in grade 3).

Conjectures:

In general, we conjectured that by surveying children across K-12 grade bands (i.e., the NCTM Principles and Standards (2000) suggests grade bands of pre-K through grade 2, grades 3-5, grades 6-8, and grades 9-12) we would find increasing coordination of

schemes for number and space, a shift in problem representation from physical models and drawings to internal representations and finally increasing levels of number fluency. In particular we expected to see a gradual increase in children's use of proportional reasoning as children develop more coordinated schemes for number and space.

We expected to find the younger children relying extensively on their physical models and drawings. Children in elementary grade-bands tend to be limited in their number sense and fluency, consequently we believed they would find few cases and have difficulty validating the relation of parts of perimeter measures to the whole. We expected that children in middle grades would exhibit greater fluency with number, greater knowledge of underlying properties of geometric figures and even classes of figures, but we expected them to struggle to coordinate number and space. In particular we expected these children to rely on drawings and physical models. Finally, we expected children in the high-school grade bands to work more abstractly (relying less on external manipulatives and drawings), and to coordinate their schemes for number closely with their schemes for space.

FINDINGS

We found three broad categories of measuring strategies among children in grade 2 through grade 10. First, we describe each category. Next, we illustrate each category with excerpts of children's responses, regardless of their grade level. Then we attend to grade-level distinctions, grouping children at grades 2 and 3, children at grades 5 and 6, and older children at grades 9 and 10. Lastly, we elaborate on more fine-grained distinctions we developed to explain differences we observed among children within the first and third categories. While a distinction between inexact and exact counting of hash marks in the middle category had already been developed in a previous study it was corroborated

here. In contrast, the distinctions we found within the first category and within the third category have only emerged in this study.

Three Strategies for Finding Perimeter and Length:

The first strategy is typified in the following composite sketch of a child responding to the rectangle task. First, a child at Level 1 bends the plastic straw into the shape of a quadrilateral. This takes some time, since the straw is difficult to form into a rectangle unless one anticipates that there will be a short-long-short-long pattern for folding the straw, and because it is difficult to hold onto three corners of a rectangle figure at once. With the interviewer helping to hold onto one or two corners, the child succeeds in forming a rectangle, or at least a quadrilateral shape.

If prompted, the child will draw and label the sides and perimeter for any rectangle drawn. When asked to check on the perimeter and the length of each of the four sides, the child writes labels beside each of the sides without making hash marks or using a ruler. These drawings at Level 1 are usually very large, approximating the actual length of the sides of the straw figure. Thus, the "measures" are typically 1-3 inches on a short side, and about 6-10 inches on a long side. When asked about the perimeter they often do not know, and struggle to sum the four reported side values on their drawing. They usually report having missed finding the perimeter of 24 as if they had just failed to locate a small missing object in a large grassy field.

Children in the first category exhibit what we call "pre-unit" strategies. These children often impose number on objects they are measuring without a specific reference to a unit. While children in this category sometimes mention inches, their use of the word is not objectively numerical, but merely subjective. Often it is based on various salient but irrelevant impressions from the task setting. Many times, children at this level

failed to use number labels to ascribe quantity to the side lengths of the rectangles and triangles they drew. When prompted for number labels, they tended to assign numbers without counting or if they counted, they counted as a way of tracking their own motions along a side of a figure they had drawn. These children tend to work in a single dimension, failing to relate length and width.

At level 2, a child typically begins by finding a rectangle with the straw, and then drawing the rectangle and marking each side with the number of hash marks to correspond to their count of notches along each edge of the straw. If prompted by the interviewer to focus on straw sections rather than the notches that separate the straw length into 24 sections, the level 2 child is often able to make a temporary and unstable shift toward counting up sections, but unable to sustain this way of counting. This child is attending to length as a quantity without having a reliable method of measuring length. Some children at level 2 are able to count the notches in exact correspondence with the straw sections, and others develop a direct way of counting sections rather than pointing to notch points along the straw's edge.

At level 3, the child is able to produce all the reasonable examples of rectangles without making hash marks along a segment, or around a perimeter. Instead, the child at level 3 can anticipate combinations of straw section lengths that would sum to 24 in a pattern of length, width, adjacent length-and then width again (L-W-L-W). Further, this child is able to employ drawings as a tool for ratios and proportions. Lastly, the child may revert to earlier, less rigorous ways of finding procedures for analyzing perimeter.

Selected Examples of Levels 1, 2 and 3

Strategy Level 1 (S2)

S2 spent the entire interview developing a single case of a triangle, the case of 11, 7 and 6. She formed the straw into this shape, and spent the remainder of the interview examining whether or not this triangle had a perimeter of 21, 23, or 24. She explored the various side length values in several ways, counting on around the vertices to get several different values on various attempts (21, 23, 24). Sometimes she included the corners or ends of the straw while counting the notches, other times she did not. When S2 was asked to explain which she thought to be the true length of the plastic straw, she responded by using the calculator and summing 21 and 24 in the following exchange:

INT: how long is the straw?

S2: 24 inches.

INT: How long is it? 21 pieces or 24 pieces?

S2: Hold up! [Now she picked up the calculator and began working. First she entered 24 and then 21 and then she paused to ask:]

Where do you get the answer?

[The interviewer showed her the "=" button and she pressed this]

S2: forty-five!

INT: why are you adding?

S2: To see if we both was wrong or right? You thought you was right!

S2 apparently guessed that the addition operation was needed to find whether the interviewer was correct to state the straw is 24 long, or whether she was correct in finding 21.

Strategy Level 1 (A2):

A2 used number labels to show the length of a perimeter by moving and writing down numbers along successive sides of a rectangle. She had been asked to show how far an imaginary ant might walk if it were to travel around the rectangle she had drawn. While moving and writing down number labels, she reached the number 24 before she had traveled all the way around the drawing of the rectangle. A2 stopped when she wrote the 24, since her goal had been to show that it would be 24 around (See Figure 1 and Figure 2).

[Insert Figures 1 and 2 about here]

Strategy Level 1 (As5):

As5 counted notches, but called them inches. She used centimeters on the ruler, but also called them inches. She did not always get the same measure with the ruler; she often miscounted the starting point. She made a 4-8-4-8 rectangle with the straw but labeled the shorter sides "3 inches" and the longer sides "8 inches" and "7 inches". She reported a perimeter of 21 (having added 3, 3, 8 and 7). Her drawings did not follow from her handling of the straw, but by a sequence of one-dimensional adjustments to the overall size of each successive rectangle shape, working gradually over and under the target perimeter of 24 until she met it (see Figure 3).

[Insert Figure 3 about here]

Strategy Level 1 (An5):

An5 worked entirely by generating sets of numbers to satisfy the algebraic expression 2L + 2W (See Figure 4). He drew successive rectangles by sketching nearly identical figures, and placing incremented number labels on the length and width of two sides of each figure. At the halfway point in his sequence of cases, he allowed the labels in his sequence to "cross" against the proportion of the drawings, matching long sides with the smaller numbers. More critically, he drew a case with a length of 12 and width 0. Note the contrast with children who identified the sequence of cases as tending towards a line segment in another study (Gagatsis & Patronis, 1990). After An5 drew and labeled the 12, 0, 12, 0 case, the interviewer asked whether one might have a length of 13. He said you could, and drew another rectangle he labeled with a -1 and 13 (See Figure 5).

[Insert Figure 4 and Figure 5 about here]

Strategy Level 2 (A3):

A3's first drawing of a straw rectangle showed the width of the straw as well as its length (see figure 6). A3 consistently counted straw pieces or centimeters when measuring for length, but was unsuccessful at coordinating the side lengths of the parts of a perimeter. Furthermore, while she was able to count straw pieces exactly as a way of measuring length, she was not able to sustain that procedure throughout the remainder of the session; she sometimes reverted to notch counting. When asked the perimeter, she reported 28 cm by summing the value of each of the sides of 8, 10, 8 and 10. She asked how it could be 28, since the length of the straw had been 24, failing to realize the need

to establish a unit of length. Still, she intended to report length as an accumulation of steps taken along the perimeter path.

[Insert Figure 6 about here]

Finally, A3 did not maintain a consistent part-whole relation between the lengths of the sides and the whole perimeter. While looking for a further case of a triangle, she overlapped the two ends of the straw and claimed that it would still be "24 long". For her, the length of the straw was conserved in spite of having overlapped the ends of the straw while forming a triangle shape.

Strategy Level 2 (J2):

J2 began the interview by counting notches to find length. But when he was prompted to look at the spaces rather than notches, he had no further trouble quantifying length for sides. He eventually found the accurate perimeter and side lengths for several figures. Furthermore, he noticed the relationship between a drawing of a 7-8-9 triangle and a straw model of a 7-7-10 triangle by explaining: "This one [pointing to the 7 side] isn't 8, and this one [pointing to the 10 side] isn't 9." The interviewer asked whether he could change the straw model in this way, and he bent one unit from the side that was 10 long and shifted it over to the adjacent side with 7, just as he had explained.

Strategy Level 2 (Se5):

Se5 drew hash marks to represent straw pieces. His marks were not evenly spaced, but he did count the pieces accurately by focusing on the spaces between the marks he

formed. He found all the rectangle cases with integer values, and extrapolated that more would exist for non-integer cases.

Strategy Level 2 (R5):

R5 did not need any scaffolding to measure with straw pieces. When asked how he found the length of a side that measured 8 straw pieces or 16 cm in length, he said: "I looked at spaces. 8." At times he allowed cases of rectangles that were not possible, such as 12, 4, 4, 4, yet when prompted, he could reject such a case as being "not one you could really have as a rectangle since its shape is wrong".

Strategy Level 2 (L10):

L10 initially drew a tiny, isosceles-shaped triangle figure and labeled it with side lengths of 20, 20 and 80 (base).

Strategy Level 3 (Se5):

Se5 used a dynamic strategy to find rectangles, integrating numeric and spatial reasoning. He said: "...you would divide 24 into 4 times and square it out, and then take a little off sides, and put it on the bottom, so it closes in as a rectangle." He made drawings of rectangles without referring to the straw. Instead, he referred to changes he could make with an internal image of the figures. For example, he began by drawing a 6, 6, 6, 6 rectangle and proceeded to draw a 5, 7, 5, 7 rectangle. While some children would do this by mere arithmetic "juggling", he was articulate about the shift of a unit from one side to the adjacent side; he dynamically "rebent" the straw as an image and expressed the results in successive drawings that exhibited careful attention to proportion.

Strategy Level 3 (J6):

J6 demonstrated a strong sense of proportional reasoning. In attempting to make drawings truly represent the straw, he defined 2 pieces of the straw to correspond to a segment of length 1 inch as a way of dilating the image to fit onto the paper space available. He easily set up and monitored this scale to translate a count of pieces to a ruler length he wanted to use with that scale. In this way, he generated all the possible integral cases of rectangles with perimeter 24. He was also able to work without a ruler, as he did when he began finding triangle cases later.

[Insert Figure 7 about here]

As he explored various arithmetic combinations for sets of 3 lengths to use to construct triangles, J6 drew some that violated the Triangle Inequality Rule (See Figure 7). However, upon reaching extreme violations of this rule he stopped himself, monitored the validity of these cases, and began rejecting cases that did not satisfy his criteria that no side could be more than half the perimeter. Furthermore, he was able to examine the relation between angles and the length of the corresponding opposite side within a triangle as a means of checking for the validity of a boundary case. While he did not reject the 5, 7, 12 triangle case, he was able to establish some of the implications of a set of cases presented by the interviewer that perturbed his image of the case and respond that this would really work "best with a rubber band". Thus, he appeared to grasp the dissonance resulting from allowing more than one shape of triangle to fit the labels 5, 7 and 12. He did reject several instances, but still allowed more than one instance of the 5, 7, 12 triangle drawing (See Figure 8).

[Insert Figure 8 about here]

Strategy Level 3 (L10):

After discussion with the interviewer on this case, L10 eventually concluded that an 80-20-20 case of a triangle could not work. Later, she concluded that the 60-30-30 case would result in a straight line rather than a triangle. She also made several other valid triangles with the straw and created a list of the valid cases. She was able to identify and reject cases of number triples that violated the Triangle Inequality.

Strategy Level 3 (Ja10):

Ja10 generated and sketched approximately 24 triangles as he responded to the task (See Figure 9). Ja10 initially struggled to decide if a 12-10-2 triangle was valid after he sketched it. He remembered the statement of the Triangle Inequality, but struggled to make it personally meaningful. After several attempts, Ja10 drew a segment with a length of 2 units, discussed that two segments of length 1 unit would meet on this segment, and described the compensatory relationship between the measure of the angles formed by the segments of a triangle and the lengths of the corresponding sides. In this way, he examined what for him was a relevant case of a triangle that would suggest the extreme implications of having a pair of sides that summed to the length of the third side, the 1, 1, 2 case. Having satisfied his need for meaning of the Triangle Inequality, Ja10 generated a nearly comprehensive list of the integer-length cases for triangles with a perimeter of 24 units.

[Insert Figure 9 about here]

Summary and Comparisons Across Grade Level Bands 2/3, 5/6, and 9/10

Grades 2 and 3: Summary and Trends

Grade 2 and 3 students exhibiting strategy level 1 and strategy level 2 usually found at least three valid cases, but no more than six cases of polygons with perimeter close to 24. Some found 9 to 10 valid cases, usually those who were able to keep an exact correspondence between notch count and the number of unitary sub-segments along the straw. The children in these grades (2-3) generally depended upon perceptual objects (either models or drawings), as a basis for generating new cases or justifying a given case. Without the straw or their drawings, they were unable to generate or analyze specific cases. Most began the interview with a notch-counting strategy in keeping with level 1, but quickly established a level 2 strategy for finding length by attending to notch marks as representatives of units (e.g. A3). C3 exhibited the widest variation in strategy: she began by counting finger movements along segments she had drawn (level 1), but later exhibited a careful iteration of a length unit to create hash marks at even intervals (finishing level 2).

While it was unusual, some students in these grades achieved exact correspondence between their measuring procedures and the actual lengths of those objects they measured. This confirmed the distinction between level 2a and level 2b described in a previous report (Barrett & Clements, (submitted for publication)). As an example, M3 counted the straw pieces accurately along the edges of each rectangle, but did not measure his drawings with a ruler. He sketched drawings that were nearly proportional sequences of rectangle shapes. Much like students in the older grades, M3 began to reason by making further sketches based on modifications to previous drawings on his paper. In fact, he came up with a triangle case that would not really work on the plastic

straw (side values of 1, 11, and 12). It was rare for a student in grades 2 and 3 to generate a case that violated the triangle inequality theorem ($a + b > c$ for all a, b, and c). L3 also violated the theorem with a triangle she had labeled to have sides of 5, 5 and 14. We will discuss her approach in a later section.

Drawings may be records of the straw count or measurable objects. Students in grades 2 and 3 used drawings in both ways, but primarily in the latter sense, as measurable objects. For example, A3 measured her own drawings in centimeters rather than writing down a count of the straw sections as a label for the sides of her drawing. Of course, these drawings were first of all records of authentic shapes they had formed with the straw, thus, by perceptual symmetry and similarity alone, the students often generated different drawings by looking at different cases they formed from the straw. Some students were attentive to intuitive, informal references to traditional units, or they employed words such as "pieces" or "parts" to identify the countable items they would use as length units. The students exhibiting level 1 strategy, as well as those we came to describe as level 2b were likely to refer to sides or the perimeter without referring either to standard units such as inches or to non-standard units such as "pieces". Few students wanted to use the ruler to make drawings. Rather, they appeared to be sketching to describe shapes they found by forming the straw. Once they formed the shapes, some like A3 were interested in finding out the length of the segments they had drawn.

These students were idiosyncratic in their ways of generating cases during the course of the interview, as shown by C3's many different ways of counting. She counted hand motions, finger taps, iterating pencil widths, finger widths, gaps between rule

marks on binder sheets, and finally, hash-marks at finger widths. Consequently, C3 was not consistent in her measures of segments, finding three completely different values in one instance (calling it 9, then 14 and then 18). Likewise, S2's inconsistent approach to finding length and perimeter also illustrates this: she counted notches without endpoints, notches with one endpoint, and notches with both endpoints at different stages of her work on the rectangle and triangle tasks.

Lastly, we note that these students were not able to generate cases of rectangles as readily as older students. It seemed noteworthy that the struggle to generate cases seemed to be one of coordination: while these students could begin with a single fold and appeared to estimate how long the first section should be, they appeared to falter at these second and later folding points when using the straw. We believe the struggle to decompose a rectangle arithmetically from its perimeter was closely tied to this physical struggle to decompose the straight straw into forms of rectangles or triangles, given that it demanded dexterity and foresight; one had to imagine the connected pattern of short-long-short-long, a pattern that appeared to frustrate their efforts at forming rectangles in particular. When asked to find triangles, they were much more successful. Hypothetically, they may have expected the folding pattern for a rectangle to be symmetrical: they often seemed to count just as far in from both ends and then try to fold this into a rectangle, meeting failure, or forming a square rectangle, but not an elongated rectangle (which they intended as a prototype). Those who began folding at the middle were more successful.

Grades 5 and 6: Summary and Trends

Students in grades 5 and 6 showed the widest range of strategy use individually and as a group. Many students began the interview at level 2a, counting notches instead of

straw pieces for length, but by the end of the interview they consistently counted pieces. Drawings may be records of the straw count or measurable objects. Students' use of drawings to record the straw count varied from using hash marks as countable items without attending to spacing between them (such as Se5) to drawing unpartitioned line segments as sides to form shapes with varying degrees of proportionality and labeling to show its length (from R5 who drew what looked like a rectangle but labeled the sides 12-4-4-4 to J6 who used proportional measures). Students at this grade level were more concerned with traditional units of measurement like centimeters or inches. Many wanted to use the ruler to draw and could not always connect straw pieces to their ruler drawn representation. D5 needed a straw to generate cases with a fixed perimeter. When asked to draw the rectangle he made with the straw, he labeled the sides with the numbers he counted. However, when he was asked to make another rectangle, he drew without using the straw and used the ruler to measure centimeters, getting a perimeter of 18. J6 also relied heavily on the ruler for drawings, but he was able to assign a unit measure to the straw pieces so he could use inches to get the correct perimeter.

Students at this grade level showed varying degrees of growth during the interview. This variation seemed to be connected to their ability to use the straw to help coordinate numeric and spatial reasoning. L5 began the interview at level 2a, measuring the length of the straw by counting notches and unable to rectify "expected" sides with straw counts. By the end of the interview, she had managed to coordinate number and space so closely that she appeared to be at level 3. Through the use of the straw to verify cases, she gained confidence in her "theory" of numbers. An5 also began the interview counting notches and had difficulty bending the straw. He gave up coordinating his numeric reasoning with the straw and allowed cases such as -1, 13, -1, 13, showing a lack

of integration with spatial images. Only when asked to draw with a ruler was An5 able to realize that his numbers did not connect to length.

Number sense of zero as it pertains to limit cases seemed to cause the most difficulty. An5 was not able to disregard the 12-0-12-0 rectangle case by drawing with a ruler. He was quite comfortable drawing a length less than one and labeling it 0. It was not until he was asked to make it with the straw that he realized it was not possible. "This one doesn't look right-it's zero-because there's no side." J6 on the other hand had enough understanding of zero length to eliminate the rectangle case, but was in conflict with the 1-11-12 triangle. When his drawing did not work, he said he just needed a smaller angle, he did not realize he needed a zero angle and what that meant for his triangle. He tried to resolve the conflict by allowing the sides to "stretch like rubber bands" so they can meet at a point above the 12 unit segment.

All students had the ability to sum the sides to find the perimeter. Most had a limited definition of rectangle that eliminated the square. Most students worked by trial and error to find rectangle sides with the correct sum for the perimeter. During the interviews Se5 and L5 progressed towards a pattern of adjusting the sides to find cases. J6 was the only student at this grade level to incorporate the dynamic effect of changing angles when changing side lengths in the triangle. Although he needed some scaffolding to recognize violations of the triangle inequality, he often measured angles to help his drawing.

Grades 9 and 10: Summary and Trends

At the high school level, we discuss the work of seven students. These students include five females and two males. They were all taking integrated mathematics courses during either ninth or tenth grade in a suburban high school in Western New York.

All seven of the students demonstrated a consistent understanding and use of length units. However, during an initial interview using only paper and pencil drawings, several students relied strictly on numerical part-whole relationships to produce examples of triangles with a fixed perimeter (in this case 120). Their drawings tended to be used as a mental prototype of a triangle with the side lengths as mere labels or placeholders rather than corresponding to the actual lengths of the sides.

The representational strategy of L10 was typical. She drew small, prototypical triangle shapes and attached numbers to each of the three side "lengths". Note the inclusion of valid cases, of cases that illustrate the equality case, and of at least one case that shows the longest side greater than the sum of the other two sides. In fact, these drawing, and similar results from other students in this secondary level group, prompted the research to conduct follow-up interviews in which the segmented straw was used to constrain the construction of potential triangles. The task was adjusted to make the fixed perimeter equal to 24 units, consistent with the actual straw.

During the follow-up interviews, each of these students was able to construct valid triangles with the straw. Further, this physical model provided the cognitive challenge to their earlier constructions, and each student reassessed the validity of these cases. Although the boundary condition of the "equality case" continued to prove troublesome for L10, she was convinced that the 80-20-20 case would not work. Later, she concluded that the 60-30-30 case would result in a straight line rather than a triangle. She was also able to make several triangles with the straw; she created a list of several valid cases. In terms of her use of length units, we place her at Level 3 based upon her ability to work with the component lengths successfully within each triangle. In addition, with the

assistance of the actual straw, she was eventually successful in producing valid triangles while eliminating those cases that violate the Triangle Inequality.

Other students in the secondary level group were able to take this reasoning with length units one step further. For example, consider Ja10 (see an overview of his work in previous section). Broadly, we classified his use of length units at Level 3. but we noted his ability to dynamically interrelate both angle and length measure in his attempt to construct meaning for the Triangle Inequality. This is not consistent with strategies of others in the grade 9/10 sample.

Considering the entire group of seven high school students, we found each student working at a Level 3 understanding of length unit. Although several students appeared to use the drawings as mere placeholders in the initial interview, the nature of the drawings appears to evolve during the second interview. Each student seemed able to retain a mental image of the triangle, with proper proportions and other important features. This mental image apparently helped them maintain reasonable relations among the sides of their triangles. We posit that students in grades 9 and 10 maintained close connections between their internal ideas for shapes and their figural representations by maintaining some degree of integrated mental image of the figures, a connection that characterized the entire secondary level group and distinguished them from their counterparts at the elementary and middle school levels.

Assuming such access to a mental image, in turn, explains how they were often able to avoid making proportional drawings while still monitoring the proportional relations among parts of figures. We believe such a mental image of a geometric shape provided

constraints to the range of cases they allowed. So, whereas middle-school students who drew non-proportional figures often allowed invalid cases, the secondary level students may have been managing a potentially non-proportional drawing by relying more on their internal image for the figure. This argument derives from Fischbein's (1993) theory of figural concepts, and supports it at once in that these findings can be explained within the theory.

Another important aspect of the secondary level students' work relates to the cognitive strategies used to produce additional examples. Most of these students worked deductively to produce all of the valid integer-length cases for the rectangles. Some of the students also worked deductively to produce most or all of the valid integer-length cases for the triangles. This tendency to work deductively, found only rarely with students at the elementary and middle grade levels, allowed these students to successfully generate multiple cases. Some of the high school students also worked inductively, particularly at the outset of each task.

Finally, the students at the high school level seemed able to connect the numerical and spatial realms. The preference for one mode over the other varied within this group; some students worked primarily with numerical cases and other students relied more heavily on drawings or actual straw constructions. However, these students were able to make meaningful connections between the numbers and the geometric figures. Ja10, for example, generated most of his valid triangle cases by varying the longest side and following a numerical pattern. At the same time, he described the relationship between the length of the different sides and the measures of the corresponding angles

in the drawn triangles. This ability to meaningfully connect the numerical and spatial realms appears to be a key factor in the demonstration of level 3 length unit strategies.

Summary Across All Students in Grades 2-10

We found the grade level performance on our task to conform broadly to our hypothesized shift: Children in grades 2 and 3 were typically shifting toward the use of an authentic unit of measure, tending to exhibit Levels 1b, 2a or 2b (see Table 1). While they still struggled to focus completely on length units being distracted by hash marks or other items along a segment they were to measure, this group of students was beginning to measure by counting hash marks as records of unitary segments along an object. Children in grades 5 and 6 were broadly capable of measurement, but they continued to struggle with the hash marks or points along an object as distractors from the iterable unit they needed. For most of these students, coordinating the sides of a figure and modifying those sides as one relates to the next figure proved too difficult. They tended to exhibit mostly level 2a and 2b strategies. While they tended to find greater percentage of valid cases than children in grades 2 or 3, they still did not coordinate both number schemes and spatial schemes as many of the children in grades 9 or 10. The children in grades 9 and 10 were mostly able to coordinate the use of a unit within a figure (level 3a), and some were able to coordinate the quantity of all 3 or 4 sides across several cases (level 3b).

(Insert Table 1 here.)

The children were categorized in keeping with the procedure described earlier in this paper, based largely on their predominant way of responding to the sub-tasks in this task protocol. We note that of the three collections of students, only grade 5 and 6 spanned 4 different strategy levels, where the older students fit within just Level 3a or Level 3b, and the younger students fit within a range from Level 1b up through Level 2b. This wide variation may follow partly from the broader range of SES for the different schools we sampled. The lower grades (2,3) involved students sampled widely as did the fifth grade students, but the students in grades 9 and 10 were all taken from a single school. Thus, we cannot argue strongly for the significance of this finding based on our sampling alone. Nevertheless, we believe that these data reflect a broad variation among fifth and sixth grade students, a variation of strategy usage and coordination of number and arithmetic schemes with a students' own spatial schemes.

DISCUSSION

Constructing Units of Length: Increasingly Coordinated Schemes for Number and Space

Consistent with our findings in a related study (Barrett & Clements, (submitted for publication)) this study supports a distinction between Level 2a and Level 2b strategies for measuring length. The capacity of some children who worked by making drawings and making hash marks to indicate and measure length along the sides of the figures varied significantly according to their understanding of length as a collection of unitary segments, or as a counting activity based on a collection of marks or points spread along the objects edges. This latter practice has been described as point counting (Cannon, 1992; Schifter & O'Brien, 1997). We believe it is more than a question of merely counting points or noticing spaces. Rather, it appears that the children who are counting points

are in a transition toward the notion that length is an accumulation of something along a dimension, while the "space counters" have actually established units of length measurement.

Our findings also suggest a level 1 strategy that does not fit well with previous descriptions of "visual guessing" as the basis for ascribing length. Children like S2. or As5 could be called "point counters", but we did not find that they were engaged in finding pseudo units of length, but rather they were counting because it is what one does to find length, and their counting was subjective and inconsistent. As5 counted centimeters, inches, and straw notches without considering whether they were commensurate or not.

Hiebert evaluated the hypothesis that Piagetian notions of conservation and length transitivity are essential for children to develop true length measurement strategies and knowledge, finding that young children could indeed carry on formal measurement without having these formal logical operations(Hiebert, 1981). While we concur that these logical operations are not necessarily precursor to path length measurement, our findings indicate that some children who are "approximating" length measurement can appear to use "inexact units" when they are merely imitating a procedure. Still, they appear unable to translate a unit along a measured object. Recall the work of S2 who merely guessed at countable items without associating her actions or words with any notion of linear extension. In listening to these children whom we classified in this study at a level 1, we found many references to standard, and non-standard units without any clear connection to iterative operations that would establish the relation between a unit, a collection of those units by iteration along a line, and the line segment that is the whole to be measured.

We found children that were classified as level 1 who counted to describe length. unlike others at level 1 who made holistic visual guesses (Clements et al., 1997, pp. 90-91). On closer analysis, we found that these "high" Level 1 children did not establish a plausible unit of length, as did other children working at level 2a or higher. We came to classify those who would make visual guesses without segmenting a length at level 1a. and those who would count to find a length without establishing a unit at level 1b (See Table 2). We focused on the intent of the child in labeling length. Those who referred to number labels as signifiers of counting and exhibited correspondence between countable items and the lengths they established, we categorized as level 1b. Those who did not establish any counting processes were described as level 1a. We noted very few instances of level 1a strategy as we analyzed these 25 interviews. Most of the children appealed to standard units of inches, or sometimes centimeters and attempted some kind of counting. Our distinction between level 1 (now level 1b) and level 2a was ultimately based on determining whether the counting strategy was used to indicate iterations or an accumulation of "sub lengths" as parts of a length. If so, we considered that level 2a (use of inexact units), suggesting the notion of iteration. If not, we considered it a "pre-unit" way of thinking about length and considered it level 1b.

We found that the development from level 2b up into level 3 was often related to the emergence of clearly proportional reasoning and to the capacity to employ multiplicative as well as additive operations. This may be similar to the ability of children to see decomposed parts of a whole, but without being able to relate them within, or beyond the whole. Tzur (1999) found children who did not carry out multiplicative translations of a unitary fraction beyond the referent whole. What Tzur calls "part-whole thinking" would correspond to the work of children at level 2b or 2a. They could use a unit to

understand a given side length, but often struggled to relate the unit beyond its side length, much less all the way around a perimeter. When children were able to relate units beyond the referent "whole" of a side to wrap all along a perimeter, we call it level 3, something Tzur calls an iterative operation.

Our findings also point to a distinction among children working at level 3. We came to distinguish a level 3a strategy from a level 3b strategy (See Table 2). We made this distinction by attending to the children's ways of handling the exhaustive challenge of finding the complete set of shapes to satisfy a specific perimeter. Some children exhibited very dynamic ways of examining specific cases, very systematic ways of generating cases, and were able to use boundary cases to reason about the completeness of their work. We categorized these children at level 3b. Others struggled when they encountered boundary cases, failed to generate as many cases, and referred to successive cases in disconnected and static ways. These were categorized as exhibiting level 3a. Both groups of children were able to monitor connections between two or more concurrent dimensions of a task, in contrast to children at level 2b who struggled to work with a second dimension. For example, R5 did not monitor the relation among "side lengths" of a rectangle labeled 12, 4, 4 and 4. When prompted, he could check it with the straw, and even rejected it before checking with the straw by criticizing its shape as "not possible". However, a child at level 3 would have been able to describe the problem by appealing to the lack of coordination. Children at level 3a or 3b rarely struggled to explain such a problem with a rectangle, though some struggled to articulate the triangle inequality.

Next we elaborate on distinctions between levels 3a and 3b, although we consider the entire range of strategy levels as a context for the discussion. The next two sections focus first on children's ways of exhausting a set of cases, and then on their ways of

examining boundary cases within a set. Finally, we conclude with a broader discussion of children's ways of coordinating representations for shapes and perimeter.

Constructing A Complete Set of Cases: Finding Cases, Generating Sequences of Cases, and Establishing Boundary Cases

We designed this study to look broadly across a wide developmental range of children as they meet an open-ended measurement problem. Furthermore, we set out to explore ways that children learn to coordinate number and space and to examine patterns as children develop increasingly sophisticated representations for quantifying spatial objects they are measuring. Thus far, we have discussed the range of strategies for unitizing length and our findings suggested a broad shift in children's coordination of number and space as they look for a set of triangles or rectangles to satisfy a fixed perimeter requirement. In this section, we examine children's reasoning processes as they generate cases, validate cases and eventually justify whether their set is complete. Our quasi-longitudinal set of observations suggests an apparent progression from experimental reasoning, to reasoning based on naïve inductive processes, and lastly deductive arguments focusing on critical boundary conditions. We also found a broad, yet consistent movement from static to dynamic reasoning processes within this sampling of children across these grades (2-10). We use this distinction between static and dynamic reasoning to distinguish between level 3a and level 3b strategies.

As a base-line example of experimental reasoning, consider S2. During the course of the 25-minute interview, S2 examined only one case, a triangle with sides of 11, 7 and 6. Initially, she said she thought of this work in terms of inches. Her response to the task consisted in finding only one case, and her effort was largely directed at validating the

case she found by folding the straw into a triangle shape. Her approach was multi-faceted, including appeals to actual notch counting (21? 23?), mental addition, calculator-assisted addition, and most of all, an appeal to the authority of the teacher/interviewer. Clearly, her attempt at this task was static, in that she never varied the case she found, nor did she proceed beyond this initial drawing and straw figure.

Next we describe the work of A3, a student who could find connections between cases she had discovered by manipulating the straw into various triangle shapes, but was never able to anticipate and construct further cases based on such natural sequences.

A3 generated drawings of two rectangles and five triangles, all cases she considered to be distinct. A3 clearly generated several cases, one quadrilateral she represented as a rectangle labeled with a perimeter of 24, and four triangles having that perimeter yet with different shapes. She used the straw to find the first few rectangles and triangles, but began making drawings after that and measuring them with a cm ruler to describe length. Interestingly, she reflected on a relation between the 10, 9, 5 triangle drawing (the third triangle she drew) and the 10, 10, 4 triangle drawing (her fourth), pointing out to the interviewer how corresponding sides were incremented by 1. This insight carried the potential for generating long sequences of cases, yet she was not able to carry this out. That same insight hints at a dynamic relation between two cases, a pattern we found increasingly prevalent among latter grades.

Next, consider the work of As5. As5 found 9 different rectangles and 2 different triangles, all with unique perimeter values. She tended to generate cases by making adjustments to a previous drawing (See Figure 3). She referred to inches as her unit to label lengths, although she measured with a centimeter scale without accuracy at times. We came to see this as "educated trial-and-error" because she kept the goal of the

perimeter and adapted successive trials based on measuring sides and summing to find whether it was 24, more or less. Furthermore, As5 showed dynamic connections between successive cases, which we believe to be based only on global features of the rectangle figures (van Hiele, 1986). For this student and the previous student, A3, the straw was useful only as an initial way of finding a shape, but not as a source of actual measures for length. Both A3 and As5 generated cases by making drawings and measuring with a centimeter ruler along the parts of their drawings. Like As5, the next student, An5, quickly abandoned the straw as he generated cases. Like many students in grades 9 and 10, he did not validate the cases he generated by measuring. But unlike those older students, he appeared to lack an internal image that would have enabled him to coordinate spatial images with his numerical scheme for satisfying a fixed perimeter.

As we reported earlier in this paper, An5 worked entirely by generating sets of numbers to satisfy the algebraic expression 2L + 2W (See Figures 4 and 5). Without spatial constraints, he allowed the introduction of cases that would not have any possible spatial referent, as when he suggested a rectangle might have a length of 12 and width 0, or when he accepted the interviewers suggestion that one might have a length of 13 and width of –1. An5 had a static mental image. For him, each rectangle does not really change in its spatial appearance. On the other hand, his work with numbers is indeed dynamic. Thus, we would characterize An5 as being in transition from static toward dynamic reasoning.

Sh5, worked with half units to find several cases of rectangles after having exhausted the integral cases, albeit she was prompted to consider a length of 2.5 by the interviewer. She did extend this line of investigation on her own to include quarter units. She was shifting toward a deductive approach (cf. Kynigos, 1993) in that she derived further

cases by referring to her relevant knowledge of fractions and decimals. She exhausted the half unit cases, thinking she would add 5 each time she would split the numbers further. We also note that Sh5 was attentive to the fit between various cases, as she expressed surprise at the combination of length 9 and width 3, being "the same" as the combination of length 9 and a half by width 2 and a half. She noted that the change in length and width was compensatory. In fact, she used this compensatory reasoning when she first checked whether the 9 and a half and 2 and a half rectangle case was valid: instead of finding the sum of 2.5, 2.5, 9.5 and 9.5, she merely noted the compensation in relation to the 9, 3 rectangle case which she already knew was a valid decomposition from 24. Finally, we believe she demonstrated dynamic reasoning by stepping through sequences of cases by a unit interval. In this section we have seen increasingly effective patterns for generating and evaluating cases of rectangles to satisfy a fixed perimeter of 24. In the following section, we examine children's gradual development of ways to handle problematic cases of triangles, by reflecting on children's ways of coordinating number schemes and spatial schemes as they examine relations among side length labels for sides of triangles (the Triangle Inequality Theorem).

Reasoning From Boundary Cases Within a Fixed Perimeter Problem: Monitoring the Triangle Inequality Relation

While this study focused directly on children's increasingly flexible, dynamic unitizations of length across grades 2-10, we found that children's developing understanding of the triangle inequality relation in the fixed perimeter tasks provided a critical window to their coordination of number and space. In this section, we discuss the work of four students, L3, J6, C9, and Ja10, representing the increasing levels of

coordination of number and space exhibited by students in this study. First we describe how we think each student understood the relation of the three sides for any triangle, and how they grappled with situations where the triangle inequality relation became problematic for them. Then we describe their reasoning as they resolved the problem, focusing on their way of coordinating number and space through various representations: using the straw model, thinking through number triples to satisfy the perimeter sum of 24, and making drawings with a ruler.

For many of the students who were in grades 2-5, the triangle inequality relation was not problematic because they consistently depended on either the straw or drawings made with a ruler to generate cases of triangles. However, some who had well-developed number sense used the straw for an initial case, and then began relying exclusively on arithmetic operations to find three-part decompositions of 24. Once they left off spatial models, they began to generate violations of the triangle theorem. The interview protocol anticipated such situations, calling for the use of the straw model or a ruler to construct the shape. This sequence of questioning was intended to help each student coordinate their spatial scheme with their already active number scheme. We begin by describing how L3 responded to this sequence of prompts having generated a 5, 5, 14 triangle case.

[Insert Figure 10 about here]

L3 generated 5 different triangle cases in all, and did not find the 5-5-14 case problematic. The interviewer first asked her how the size of the number labels related to the length of each side in the figure she had drawn (see Figure 10). She responded by drawing this triangle case again without labels, showing a mostly equilateral shaped

triangle (see the top right triangle in Figure 10). Then the interviewer asked her to use a ruler, and she found she could not close the triangle using standard centimeter units. But she did not conclude from this that the triangle case was impossible. Instead, she said she could fix this problem if she were to use inches and centimeters (See Figure 11). She "fixed" the problem by drawing a triangle measuring roughly 5 inches by 5 inches by 14 centimeters (14 at its base).

[Insert Figure 11 about here]

We interpret her work as an example of how students may use their strong number sense to analyze an open-ended measurement task without realizing the importance of spatial models to complement such analysis (Barrett & Clements, 1996). We believe L3 would have needed further opportunities to see the implications of letting go of the connection between number and space on this kind of task before she would develop an understanding of the triangle inequality. We do not believe she grasped the critical relation of three segments forming a triangle because she did not monitor her violation of the inequality and because she interpreted the interviewers' prompts on this case (5-5-14) as a critique of her use of proportion rather than concluding that the case was not possible.

We continue our discussion of students' increasing levels of coordination of number and space in responding to problematic instances of the triangle inequality by examining the work of J6. Like L3, J6 eventually abandoned his use of the straw and his use of a ruler to rely exclusively on his number sense to find number triples. In contrast to L3, J6 was more systematic as he held one side constant and allowed the other two to vary in a stepwise pattern. Later in the interview, J6 began making small, but roughly proportional

triangle sketches to display each successive set of three number labels. Furthermore, he appeared to reflect on a mental image to show changes on each new triangle drawing as he changed the numbers to find the next case.

J6 first allowed a violation of the triangle inequality when he drew a 16, 3, 5 case. He remarked, "I am off in the design of it, but it adds up to 24". We believe he was prompted by spatial reasoning to question the case, although he decided to leave it. He moved directly to consider the case of 22, 1 and 1. We think he broke his systematic generation of cases, to move directly to what he suspected would be problematic. He sketched a long segment labeled 22 and attached a short segment he labeled 2, saying, "Can I check this one on the straw". After working with the straw, he said this would not work and went back to the 16, 3, 5 case to cross it out. He explained, "I don't think that one line can be more than one half of the total perimeter of the triangle." When J6 noted the problem with the "design" of the 16, 3, 5 drawing, we suspect he compared his drawing to an internal image that conflicted with his drawing. He may have seen the disproportion by relating the 3 as a unit and questioning how 5 of these would fit into a length of 16 as drawn (see figure 8).

More broadly, we note J6 as one who kept his number sense in check with spatial drawings, even in the absence of external models. He depends on external models as proof of his theoretical claim that one side cannot be more than half the perimeter. Recall also that J6 was unable to finally resolve the problem of a boundary case for the triangle inequality (see his work with the 5-7-12 case) since the plastic straw and making drawings with rulers are both measurement processes that are all ultimately imprecise. Students in grades 9 and 10 exhibited similar struggles with cases of numbers that violated the triangle inequality, but unlike J6, they were often less inclined or less

concerned about checking their ideas by referring to a physical model such as the straw, or by looking at a ruler.

C9 required extensive prompting to consider issues around the triangle inequality. When prompted to measure out specific cases, she found that some could not be drawn (she could not draw a 20, 30, 70 case), while others could be drawn (e.g., she drew a 50, 50, 20 case). Eventually, she came to depend on a notion that "these two have to equal that one", meaning that she recognized the impossibility of having two sides that would not reach to the length of a third side. However, in contrast to students in earlier grades, C9 represents the secondary students tendency of working from mere sketches alone without referring to a ruler or to a physical model. This is both an advantage and a disadvantage: while the students in grades 9 and 10 discovered more cases more often and were mostly successful in describing the complete set of integral cases for rectangles and for triangles, they were also more likely to produce and allow invalid cases of triangles. Furthermore, even when prompted to consider boundary cases or extreme violations, some students struggled to describe a rule for later reference.

A few students at grade 10 exhibited comprehensive coordination of space and number as they employed a constant monitor for relating the side lengths to one another as they developed sequences of cases. Ja10 proposed several cases of number triples that might form triangles, but he rejected those invalid cases as he considered each new case in turn. Furthermore, he came to explain the triangle inequality by relating side length to the size of the opposite angles, and by maintaining the relation among the three sides. His explanation was based on the case of 1, 1, 2. He eventually argued that the angles must get larger as one pulls the 1-1 vertex away from the midpoint of the side labeled 2, and that means that those sides labeled 1 must now grow larger, since the angles are

growing larger. This was the most advanced instance of coordinated images we identified in this study.

Ja10 appears to surpass the reasoning of J6 in that Ja10 purposely chose to argue from the boundary condition rather than a less extreme violation such as the 1,22, 1 case used by J6. Ja10 also reasons without relying on a physical model, but argues on the basis of a dynamic relation among various critical components of the triangle. This exemplifies Fischbein's emphasis on the need to integrate figure and concept into a third, distinct representation if children are to forge flexible, transferable knowledge of geometric entities (1993). Ja10 exhibited a figural concept of a triangle that could be foreshadowed by an external figure (Ja10 draws and points to the drawing of the 1.1.2 case), but one that derived its force by enacting the dynamic interplay among parts of an internal image (Goldenberg, Cuoco, & Mark, 1998; Olive, 1998), something Lehrer has recently pointed to as reasoning based on a process of "morphing" (Lehrer, Jenkins, & Osana, 1998).

We claim the power of his argument came from his reliance on a dynamic internal image since he depended solely on a narration of how one might modify the parts of the external figure (the line segments on his drawing) and the effect that would have on the various parts of the figure as an approximation of a triangle. Further, we argue that the reasoning of Ja10 in the context of this task highlights the importance of prompting and guiding a student toward elaboration of a conjecture and the development of argumentation that can securely justify the claim by a careful selection of cases and by linking the cases together (Koedinger, 1998, p. 344-346). In this situation, Ja10 used "model-based investigation" to elaborate on the critical boundary condition using a "model-and-measure" strategy that was dynamic in nature.

This broad survey of several cases of children's ways of meeting problematic instances of the triangle inequality has highlighted the increasingly coordinated schemes for number and space exhibited by children throughout our cross-sectional sample from grades 2-10. Clearly, greater coordination of children's own number and spatial schemes supported increasingly sophisticated knowledge of the triangle inequality. We believe children's construction of their own connections between the two schemes played significantly into their capacity to monitor and extend their set of valid cases. While their were strong developmental patterns of increasingly abstract images for the figures across the range of ages for children in our study, there were notable anomalies: there were children that were "free" to disconnect number and space who struggled to regain a clear connection between the two schemes when prompted (e.g. A5, who went as far as the case for 12, 0, 12, 0 and even to allow a rectangle of –1, 13, -1, 13), while there were others who appeared to respond to prompting by forging a greater connection among their schemes (e.g. C3, who built up increasingly robust counting schemes for measuring length when prompted to restate a given length). These conclusions support arguments for instruction aimed at prompting children to forge connections among representations for themselves in contrast to instruction that provides "pre-connected" representations (c.f. Bowers, 1997; Clements, Battista, & Sarama, 1995; Olive & Steffe, 1994; Steffe & Olive, 1996).

Coordinating Representations for Path Length: Developmental Shifts

In summary, we found children moving broadly from a complete reliance on physical figures, either the straw as a shape or the drawings the children made by recalling prototypical shapes, to a readiness to connect drawings with the straw, or with

their internal images for shapes. For children in the early grades, and for some in the middle grades, there was a strong inclination to work exclusively with the straw as a source of shapes (if one can make a shape of a rectangle with the straw, then that particular case "comes" into existence). Most of these could also work by making drawings and then measuring them. They did not anticipate the measures of the lengths of the straw figures they formed, or of the drawings they made (Consider the work of C3. or S2). Rather, once formed or drawn, the child would then measure and "discover" the length of each of the sides. Further, they often found sums that did not conserve the overall length of the straw without noting or reflecting on losses or gains in overall path length. There were exceptions, such as L3, who worked by subdividing 24 and generating side values.

Children in the middle grade band tended to generate several figural cases of rectangles using the straw as a referent. Their work focused on drawing records of cases they discovered by forming the straw into those shapes, or by making drawings and occasionally checking these against the straw (Recall the successes of Se5 as he counted spaces along the straw and recorded various shapes). They expected conservation. and worked to find ways of accounting for all 24 units of length along the 3 or 4 sides of a shape they formed with the straw. These children typically did not violate the triangle inequality relation because they worked closely with the straw (as when J6 asked to refer again to the straw, and verified his concern that some triangles would not work by trying to use the straw to form them). One child, L5, exhausted the set of rectangles having integral side measures and found most of the triangles while some of the high school children did not find so many cases. We believe her "measurement fluency" for perimeter and shape explorations followed from her commitment to the straw as a

referent for validating arithmetic combinations she found by incrementing successive side lengths.

Finally, the high school students were often inclined toward strictly numerical operations with three side lengths (or four for rectangles). They did not often refer to the straw after forming the initial case required in the protocol. They also tended to use drawings that were roughly proportional or even drawings that were "not drawn to scale" as they had often met in their texts and standard tests. For those who were able to maintain and operate on an internally consistent mental image of the shapes, this strategy was efficient, and open to monitoring for proportionality. Thus, these students were apt to catch their own violations of the triangle inequality, or even drawings that were lacking proportionality (Consider the work of Ja10 in identifying the problem with 2, 10 and 12 as a triangle case). However, for some, this strategy of drawing shapes that were sometimes no more than placeholders was not conducive to representations of a connection between number values and spatial lengths. Given this disconnection, these students often failed to monitor or identify violations of the triangle inequality (consider L10 in proposing a triangle with sides of 20, 20 and 80).

CONCLUSIONS AND IMPLICATIONS

In conclusion, we found support for our initial conjectures that children's reasoning about length and perimeter develops through three types of strategy, suggesting a correlation to their broader development of increasingly coordinated knowledge of number and space. However, we noted distinctions among children at the first level as they tend to refer to counting as a way of establishing quantity for length or not, and distinctions among children at the third level as they tend to employ more or less

dynamic internal images and more or less connected sequences of argumentation in response to such a broad-ranging task. We found that children's capacity for making drawings that establish connections between number and space (Outhred & Mitchelmore, 2000, p. 164), for employing measuring methods that coordinate length units, side lengths, and perimeter as both a sum of sides and accumulation of path units(Clements, Battista, & Sarama, 1998), and lastly, for arguing across close sequences of cases of related shapes (Lehrer et al., 1998) constituted a necessary (and possibly sufficient) set of practices and ideas for responding to an open-ended perimeter measure task. Furthermore, our examination of children's reasoning patterns suggest a developmental trajectory for children's capacity to argue and justify their thinking about measurement: initially, children were inclined to find an example and were content with a single example, perhaps an existence proof rather than a response to the challenge of exploring an entire set. That is, they struggled to grasp the problem of exhausting a set. Later, children were more likely to exhaust a finite set of integral cases, but failed to initiate cases besides the integer-valued measures of side length. Perhaps there is a need to learn more how children come to relate the density of the number line in relation to the density of real space along any dimension (cf. Pettito, 1990).

This study suggests that children in elementary grades may gain by learning to consider more than one case, and by being guided to see several cases of a situation. The children at the middle grades seemed to be most adept at reasoning on this task when they were able to vary the coordination between their number and space schemes for perimeter. The middle-school children who generated the most cases usually allowed themselves to work apart from checking every case with the straw. Rather, they checked

occasionally, and depended on proportional drawings to maintain validity at other times. The high school aged children exhibited wide-ranging ability to understand the boundary conditions. This appears to be a productive area of questioning. For all the grade bands it appears that teachers must cope with extremely diverse understanding and strategy development. But there were clear delineations between the grade bands indicating that children in grades 2 and 3 are centrally focused around the issue of units and iterating units, while the children in grades 5 and 6 were able to deal productively with issues of generalization, as long as they could maintain a reference to some physical representation. The students at grades 9 and 10 appeared to spend most of their effort grappling with boundary conditions and learning to extend cases that would examine the density and continuous characteristic of length measurement.

These results also argue for increasing the use of drawings that children construct entirely on their own, beginning on blank paper to allow them to grapple with constructing units, and establishing proportional relations they notice. The children in this study who exhibited coordinated reasoning between number and space consistently produced the most sophisticated responses to the task, but it seems that allowing children to dissociate spatial length from arithmetic procedures was a way of generating further cases, or possible cases. The varying role of the interviewer in allowing inconsistent or impossible cases to come out and be considered at times, and to press children toward extremely implausible cases often led the children to construct more rigorous internalized knowledge of critical geometric relations (like the triangle inequality, or opposite sides of a rectangle are equal) suggests that teachers may support logical argumentation processes by helping children meet such extreme cases as to be

able to return to less extreme cases with an improved connection between their number and spatial schemes.

Finally, we found the framework of levels (see Table 2) a critical tool in understanding children's increasingly sophisticated measurement strategies and knowledge. By using this tool to promote reasoning trajectories, to assess children's strategy development, and to understand children's ways of understanding length, we believe teachers can make better decisions about their instruction to guide children as they learn to measure length and perimeter. By learning how children think about measuring length and perimeter at different levels of development, teachers can attend more productively to children's needs to establish units, to relate space to their knowledge of addition and multiplication operations, to connect proportional reasoning to experiences with measurement and to the use of drawings that will support connections and coordination of units and collections of units of length. While some studies have begun to describe the process of translating these accounts of children's developing knowledge for geometry and space into classroom environments by working directly with classroom teachers (Barrett & Clements, (submitted for publication); Clements et al., 1998; Lehrer et al., 1999) there is a need for careful accounts of teacher's struggles as they work to translate their knowledge of children's thinking about measurement into their teaching practices.

Table 1. Summary of Children's Strategy Levels On Perimeter Task: Grade Level versus Strategy Level

	GRADES 2-3	GRADES 5-6	GRADES 9-10
Strategy 1A: no use of units; assign numbers by guessing			
Strategy 1B: measure by counting while scanning or moving	G2, A2, S2	An5	
Strategy 2A: selects an item with a length feature but iteration allows gaps or overlaps	D3, A3, C3	D5, C5, As5, R5, T5	
Strategy 2B: iteration of unit to show length along 1 line	L3, J2, M3	Se5, L5, Sh5	
Strategy 3A: coordinated unit within 1figure		J6	J9, T10, N10, C9, L10
Strategy 3B: coordinated unit among and between several cases/figures			Ja10, S10

Table 2: Levels of Length Strategy

Level	Attributes
1a: no units	Does not use units, only holistic or direct comparison. May assign number values by guessing.
1b: no units, but tries to count	Attempts to validate measurement by counting while scanning or moving. May assign number by guessing or estimation.
2a: poorly iterated unit items	Selects an item with a length feature, but does not structure the iteration operation to prevent gaps/overlaps
2b: iterates a single sequence	Good correspondence between iterated unit items and collection to show overall length on a single line
3a: iterates collections of units as a Coordinate Unit, within a single figure	Can operate with a collection of units and move from partitioning to grouping and back and forth. But, does not integrate whole sides with other sides with entire perimeter, beginning from primary units
3b: iterates and coordinates units and coordinate units among and between several cases of figures.	Iterates coordinate units and units. Can coordinate operations between collections of figures. Can operate dynamically among coordinate units across parts of figures and across figures.

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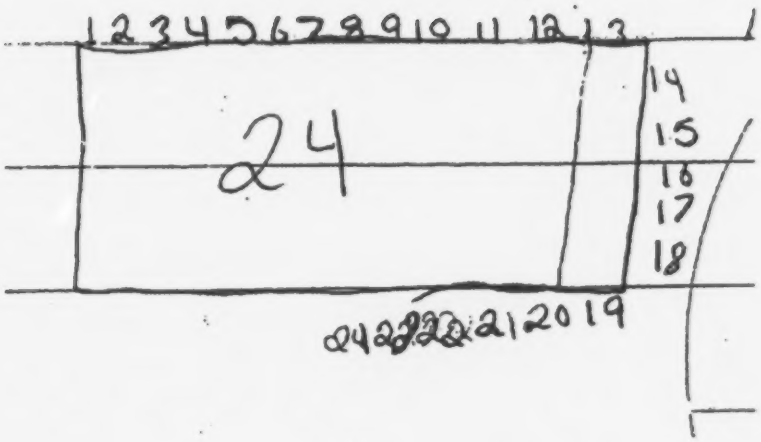

Figure 1. Student A2 distributed number labels while moving around the perimeter of her prototypical rectangle.

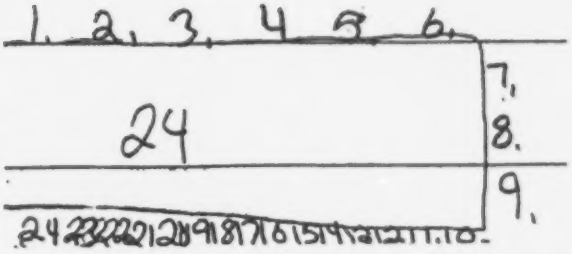

Figure 2. Student A2 distributed "length" differently as she approached the end of the perimeter (so she thought) along the third section of the perimeter. She was using the edge of her paper to show the fourth side of this rectangular figure.

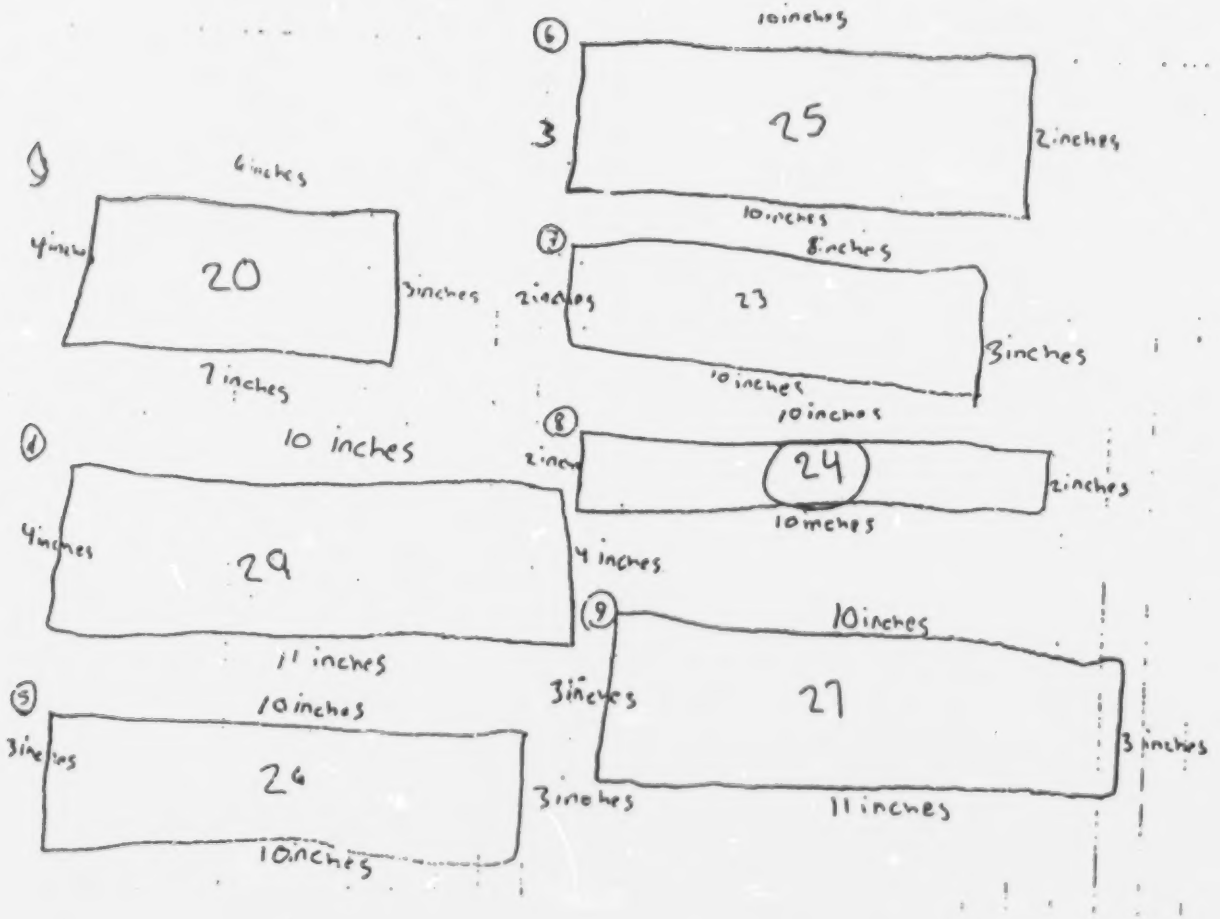

Figure 3. Student AS2 drew and then measured along successive figures to find a rectangle with perimeter 24.

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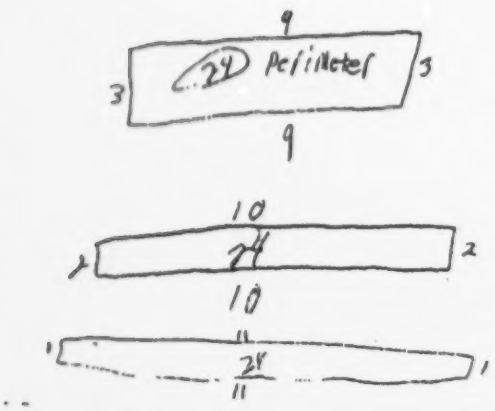

Figure 4a. Student An5 began drawing further cases of rectangles that followed from the 9, 3, 9, 3 case he had formed with the plastic straw. His next two cases: 2-10 and 1-11.

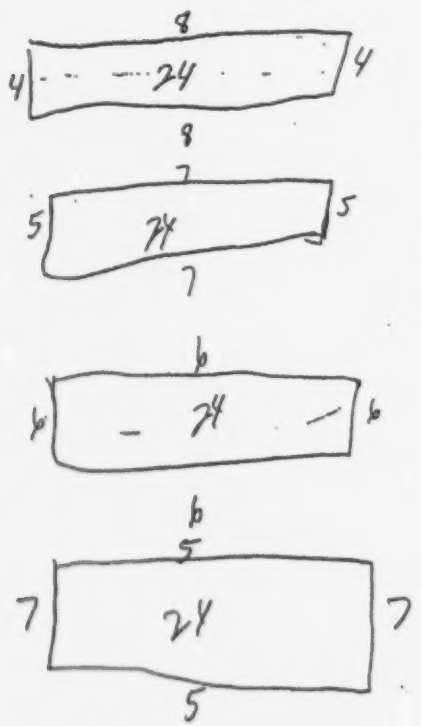

Figure 4b. Student An5 continued, drawing the next four figures on another page.

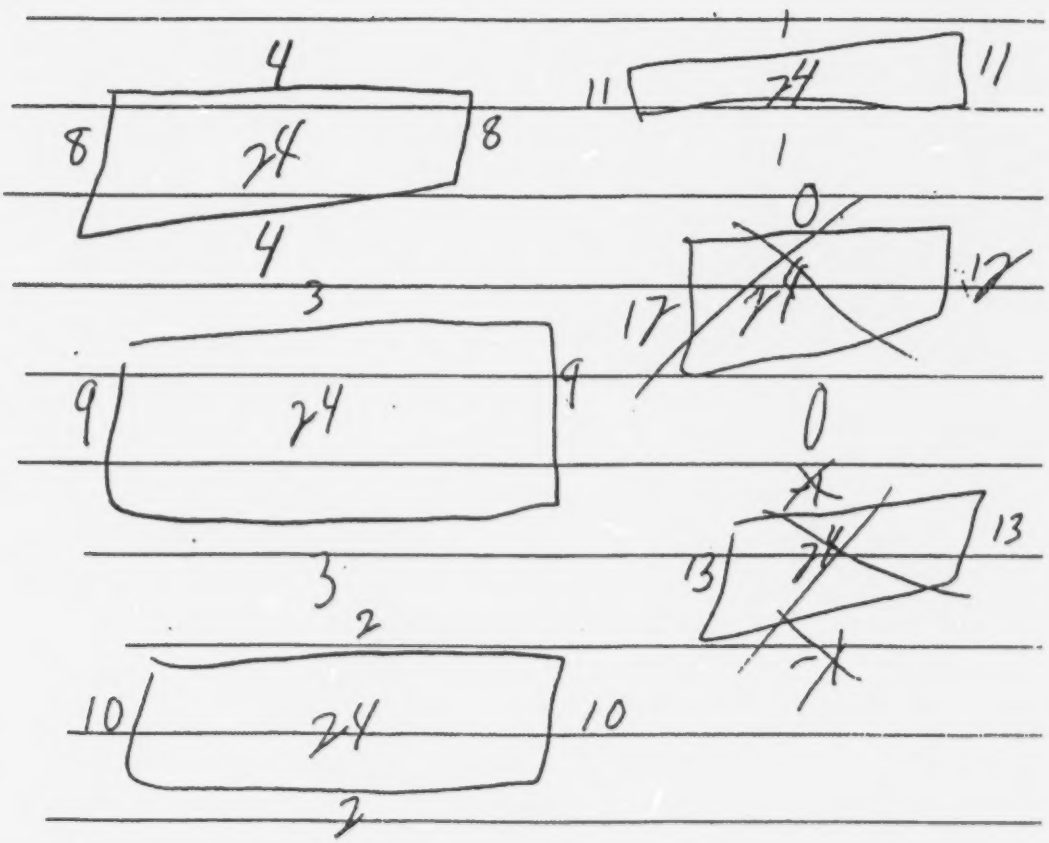

Figure 5. Student An5 completed his description of possible rectangles by drawing the first five figures. When asked "Can there be one with length 13?" he drew the sixth figure (the 13, -1, 13, -1 case). After several minutes of discussion, he crossed out the last two cases.

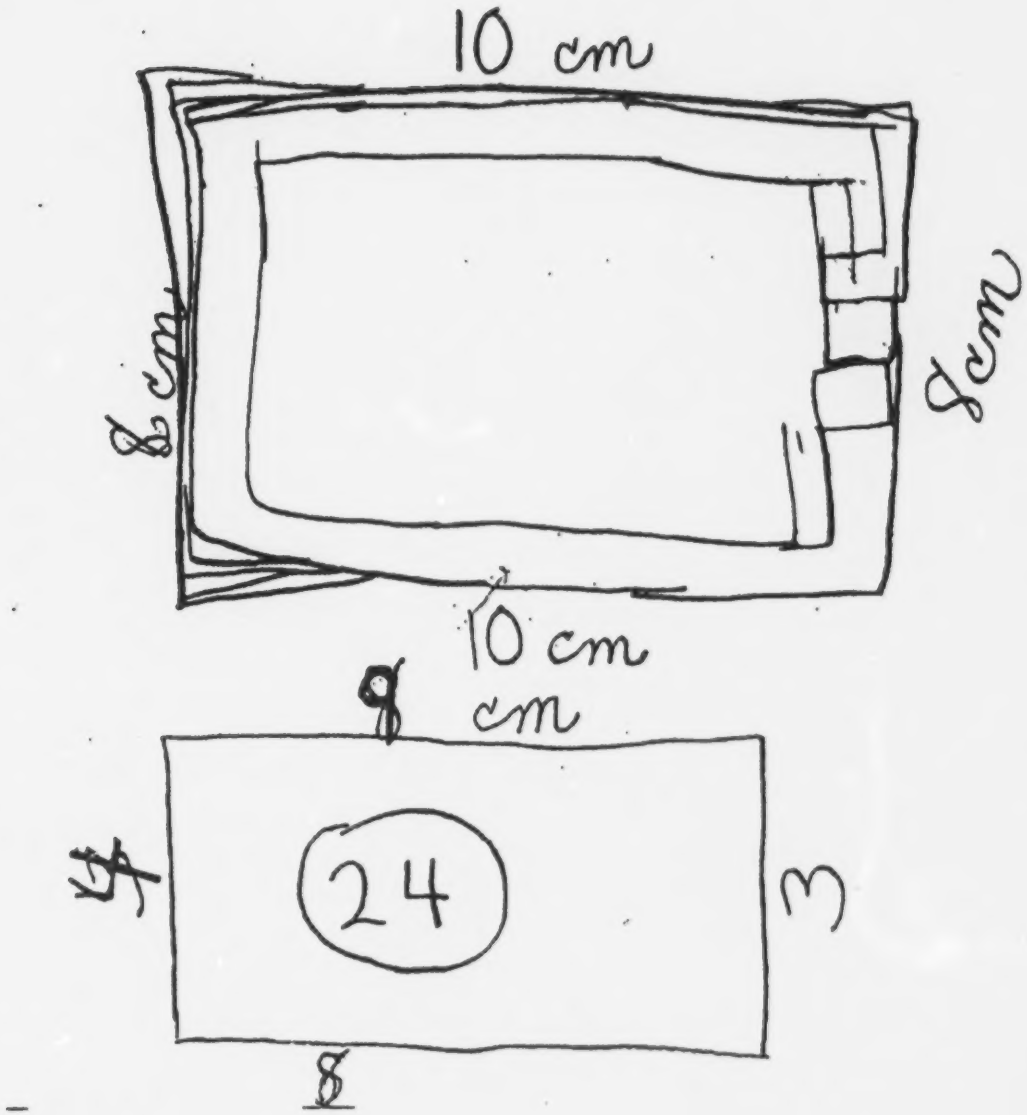

Figure 6. Student A3 began by drawing the upper figure to portray extraneous attributes of the plastic straw figure used to find rectangle cases. The lower drawing dispenses with those other attributes (straw thickness, and portraying notches along the straw).

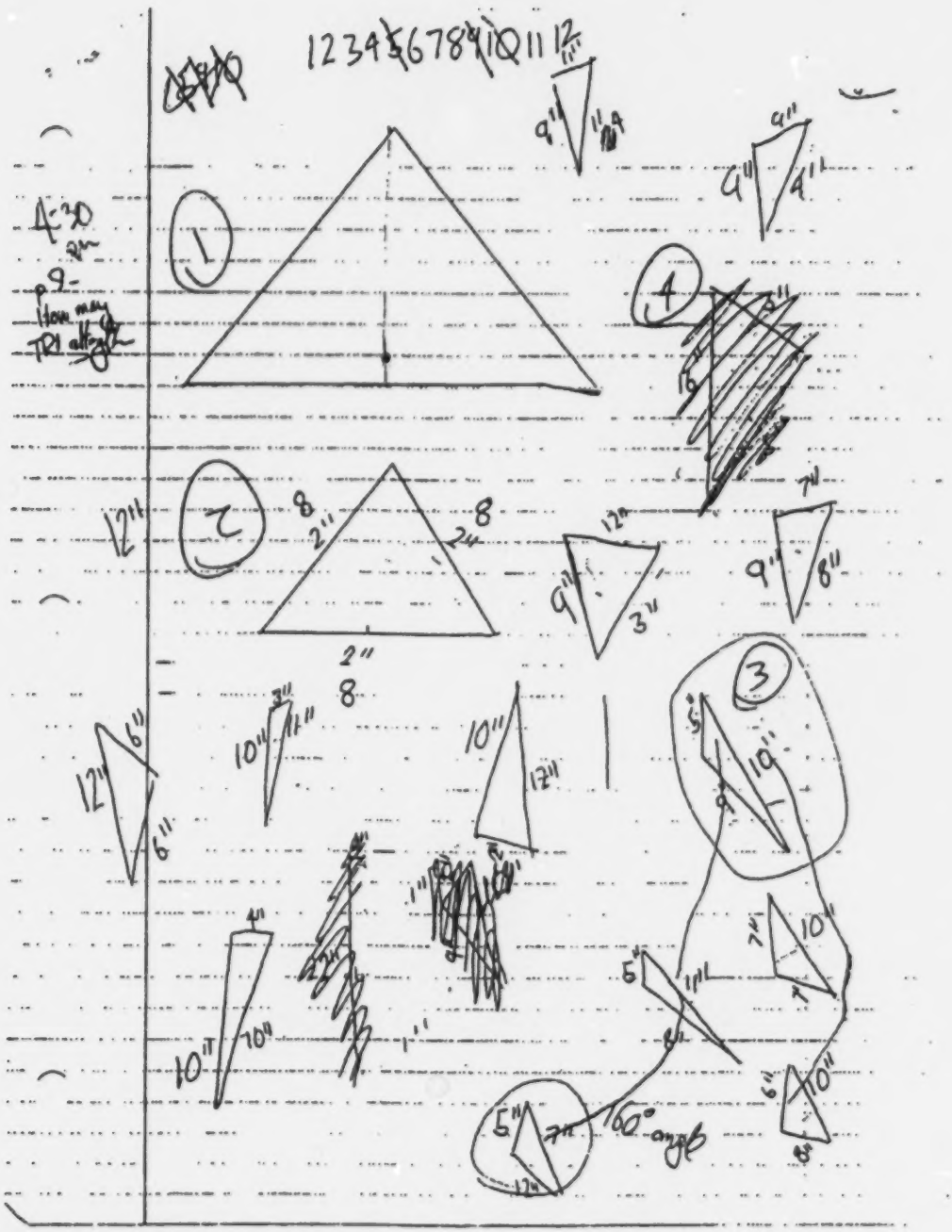

Figure 7. Student I6 examined the set of possible triangles given the perimeter of 24. The labels 1, 2, 3 and 4 describe the first four cases drawn by the student.

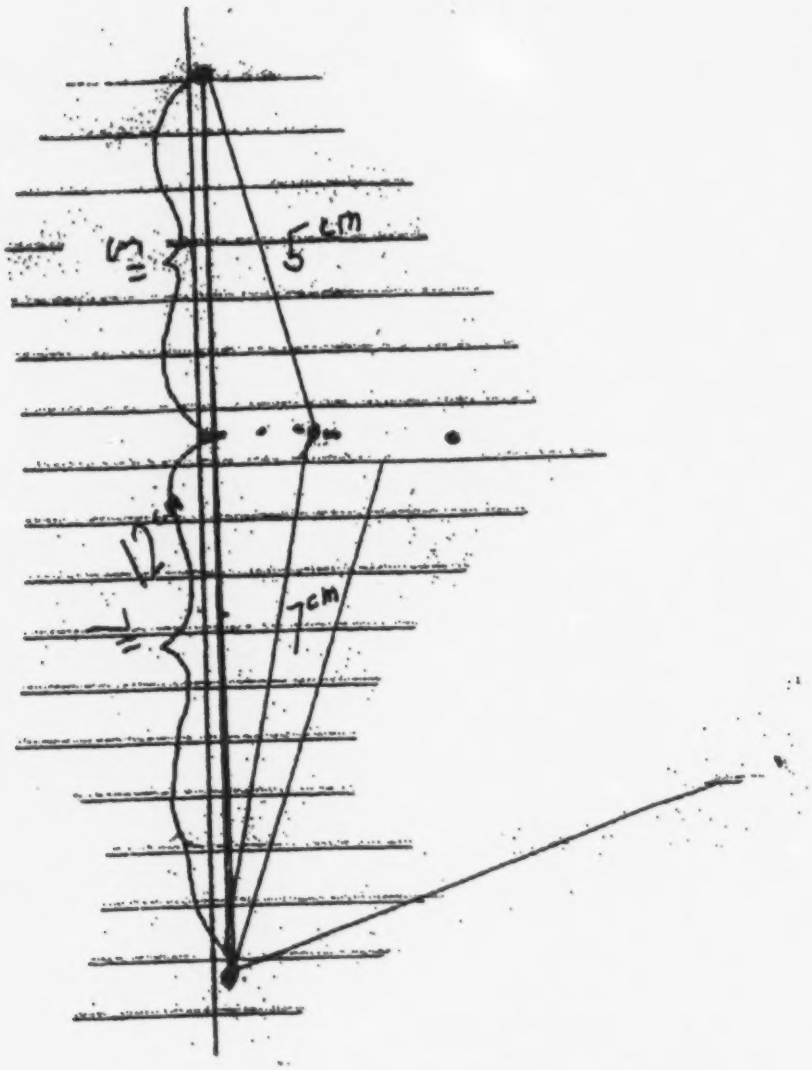

Figure 8. Student J6 used a ruler to draw the triangle case with sides of 5, 7 and 12.

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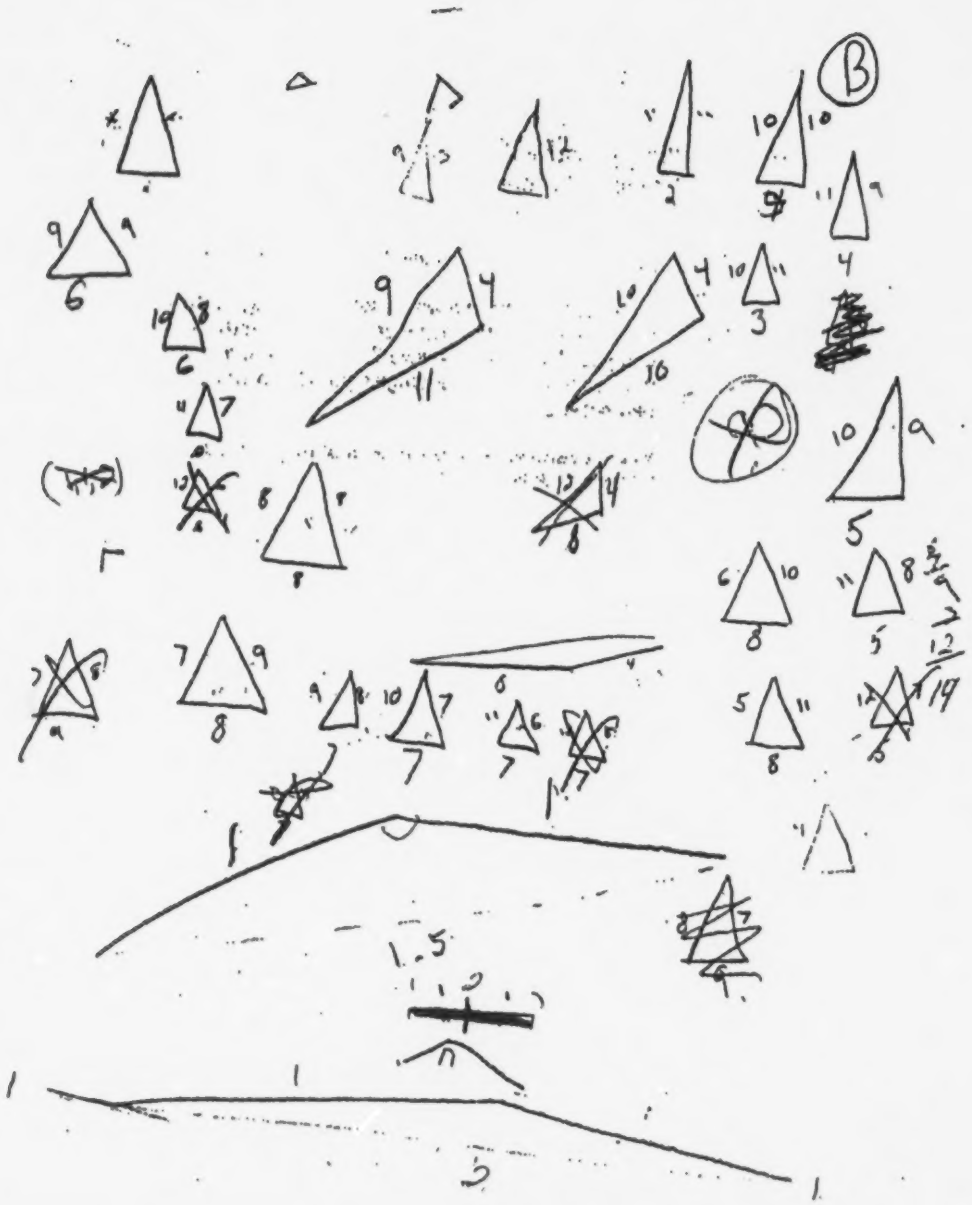

Figure 9. Student Ja10 explored all possible triangles and examined the two cases 1, 1, 2 and 1, 1, 1.5. He used these last two triangle cases to explain the rule for relating the sides of triangles (see lower section of the figure).

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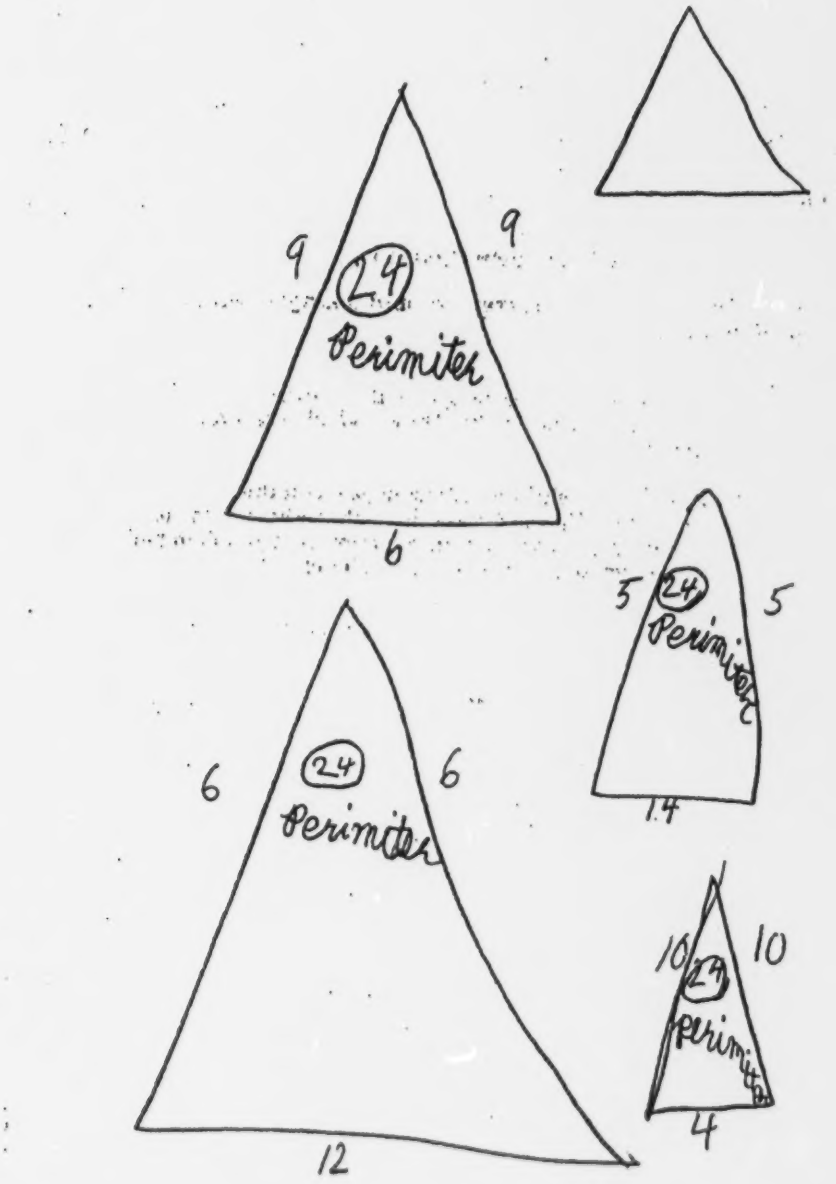

Figure 10. Student L3 explored possible cases of triangles with perimeter 24. The upper right triangle was drawn after the 5, 5, 14 as a revision of that figure.

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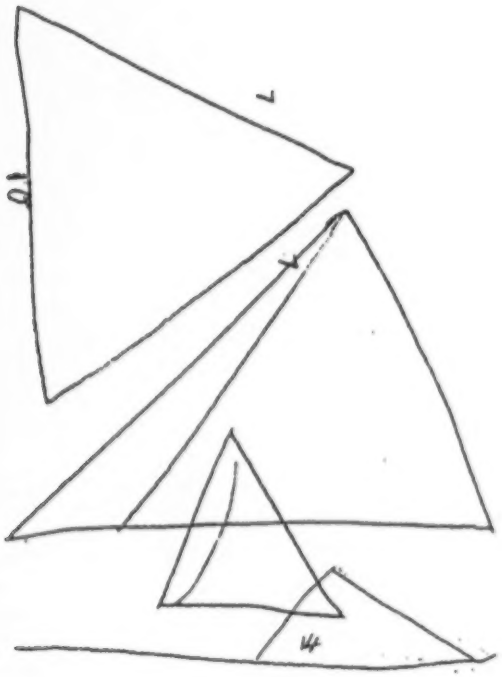

Figure 11. Student L3 struggled to use a ruler with both a centimeter scale and an inch scale to portray the triangle case with sides of 5, 5 and 14. She drew the center triangle by mixing the two scales, producing a figure with sides of 5 inches and a base of 14 centimeters.